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2006 J. Phys. A: Math. Gen. 39 9839

(http://iopscience.iop.org/0305-4470/39/31/013)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 9839–9844 doi:10.1088/0305-4470/39/31/013

General pure multipartite entangled states and the Segre variety

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Received 15 March 2006, in final form 22 June 2006
Published 19 July 2006
Online at stacks.iop.org/JPhysA/39/9839

Abstract
We construct a measure of entanglement by generalizing the quadric polynomial of the Segre variety for general multipartite states. This measure of entanglement works for any pure state and vanishes on multipartite product states. We give explicit expressions for general pure three-partite and four-partite states, and compare our measure of entanglement with the generalized concurrence.

PACS numbers: 03.67.Mn, 42.50.Dv, 42.50.Hz

1. Introduction

Quantum entanglement has received a lot of attention in recent years because of its usefulness in many quantum information and communication tasks such as quantum teleportation and quantum cryptography. However, there are still many open questions concerning the quantification and classification of multipartite states and also their true nature. Thus a deep understanding of this interesting quantum mechanical phenomena could result in the construction of new algorithms and protocols for quantum information processing. One widely used measure of entanglement is the so-called concurrence [1]. Segre embedding provides a setup for geometrical construction of concurrence [2, 3]. In this paper we will expand our result on the Segre variety [4], which is a quadric (zero locus of quadratic polynomials), by constructing a measure of entanglement for general pure multipartite states, which also coincides with concurrence of general pure bipartite and three-partite states [5].

Denote a general, composite quantum system with m subsystems as $\mathcal{Q} = \mathcal{Q}_m^p(N_1, N_2, \ldots, N_m) = \mathcal{Q}_1\mathcal{Q}_2\cdots\mathcal{Q}_m$, with the pure state $|\Psi\rangle = \sum_{k_1=1}^{N_1}\sum_{k_2=1}^{N_2}\cdots\sum_{k_m=1}^{N_m}\alpha_{k_1k_2\ldots k_m}|k_1\rangle\otimes|k_2\rangle\otimes\cdots\otimes|k_m\rangle$ and corresponding to the Hilbert space $\mathcal{H}_{\mathcal{Q}} = \mathcal{H}_{\mathcal{Q}_1}\otimes\mathcal{H}_{\mathcal{Q}_2}\otimes\cdots\otimes\mathcal{H}_{\mathcal{Q}_m}$, where the dimension of the jth Hilbert space is $N_j = \dim(\mathcal{H}_{\mathcal{Q}_j})$. We are going to use this notation throughout this paper. In particular, we denote a pure two-qubit state by $\mathcal{Q}_2^p(2, 2)$. Next, let $\rho_{\mathcal{Q}}$ denote a density operator acting on $\mathcal{H}_{\mathcal{Q}}$. The density operator $\rho_{\mathcal{Q}}$ is said to be fully separable, which we will denote by $\rho_{\mathcal{Q}}^{\mathrm{sep}}$, with respect to the Hilbert space decomposition, if it can be written as $\rho_{\mathcal{Q}}^{\mathrm{sep}} = \sum_{k=1}^{N} p_k \bigotimes_{j=1}^{m} \rho_{\mathcal{Q}_j}^{k}$, $\sum_{k=1}^{N} p_k = 1$ for some positive integer N, where p_k are positive real numbers and $\rho_{\mathcal{Q}_j}^{k}$ denotes a density operator on Hilbert space $\mathcal{H}_{\mathcal{Q}_j}$. If $\rho_{\mathcal{Q}}^{p}$ represents a pure state, then the quantum system is fully separable if $\rho_{\mathcal{Q}}^{p}$ can be written as $\rho_{\mathcal{Q}}^{\mathrm{sep}} = \bigotimes_{j=1}^{m} \rho_{\mathcal{Q}_j}$, where $\rho_{\mathcal{Q}_j}$ is the density operator on $\mathcal{H}_{\mathcal{Q}_j}$. If a state is not separable, then it is said to be an entangled state.

Here are some prerequisites on projective algebraic geometry [6, 7]. Let $C[z] = C[z_1, z_2, \ldots, z_n]$ denote the polynomial algebra in n variables with complex coefficients. Then, given a set of q polynomials $\{g_1, g_2, \ldots, g_q\}$ with $g_i \in C[z]$, we define a complex affine variety as

$$\mathcal{V}_{\mathbf{C}}(g_1, g_2, \ldots, g_q) = \{P \in \mathbf{C}^n : g_i(P) = 0, \forall\, 1 \leqslant i \leqslant q\}, \tag{1}$$

where $P = (a_1, a_2, \ldots, a_n)$ is called a point of $\mathbf{C}^n$ and the a_i are called the coordinates of P. A complex projective space $\mathbf{CP}^n$ is defined to be the set of lines through the origin in $\mathbf{C}^{n+1}$, that is,

$$\mathbf{CP}^n = \frac{\mathbf{C}^{n+1} - 0}{(x_1, \ldots, x_{n+1}) \sim (y_1, \ldots, y_{n+1})}, \qquad \lambda \in \mathbf{C} - 0, \quad y_i = \lambda x_i \quad \forall\, 0 \leqslant i \leqslant n+1. \tag{2}$$

Given a set of homogeneous polynomials $\{g_1, g_2, \ldots, g_q\}$ with $g_i \in C[z]$, we define a complex projective variety as

$$\mathcal{V}(g_1, \ldots, g_q) = \{O \in \mathbf{CP}^n : g_i(O) = 0, \forall\, 1 \leqslant i \leqslant q\}, \tag{3}$$

where $O = [a_1, a_2, \ldots, a_{n+1}]$ denotes the equivalent class of point $\{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\} \in \mathbf{C}^{n+1}$. We can view the affine complex variety $\mathcal{V}_{\mathbf{C}}(g_1, g_2, \ldots, g_q) \subset \mathbf{C}^{n+1}$ as a complex cone over the complex projective variety $\mathcal{V}(g_1, g_2, \ldots, g_q)$.

2. Multi-projective variety and a multipartite entanglement measure

In this section, we will review the construction of the Segre variety. Then, we will construct a measure of entanglement for general multipartite states based on a modification of the definition of the Segre variety. This is an extension of our previous result on the construction of a measure of entanglement for general pure multipartite states [4]. We can map the product of spaces $\mathbf{CP}^{N_1-1} \times \mathbf{CP}^{N_2-1} \times \cdots \times \mathbf{CP}^{N_m-1}$ into a projective space by its Segre embedding as follows. Let $\left(\alpha_1^i, \alpha_2^i, \ldots, \alpha_{N_i}^i\right)$ be points defined on the complex projective space $\mathbf{CP}^{N_i-1}$. Then the Segre map

$$\begin{aligned}\mathcal{S}_{N_1,\ldots,N_m} : \mathbf{CP}^{N_1-1} \times \mathbf{CP}^{N_2-1} \times \cdots \times \mathbf{CP}^{N_m-1} &\longrightarrow \mathbf{CP}^{N_1 N_2 \cdots N_m - 1} \\ \left(\left(\alpha_1^1, \alpha_2^1, \ldots, \alpha_{N_1}^1\right), \ldots, \left(\alpha_1^m, \alpha_2^m, \ldots, \alpha_{N_m}^m\right)\right) &\longmapsto \left(\alpha_{i_1}^1 \alpha_{i_2}^2 \cdots \alpha_{i_m}^m\right).\end{aligned} \tag{4}$$

Next, let $\alpha_{i_1 i_2 \cdots i_m}$, $1 \leqslant i_j \leqslant N_j$ be a homogeneous coordinate function on $\mathbf{CP}^{N_1 N_2 \cdots N_m - 1}$. Moreover, let us consider the composite quantum system $\mathcal{Q}_m^p(N_1, N_2, \ldots, N_m)$ and let $\mathcal{A} = \left(\alpha_{i_1 i_2 \ldots i_m}\right)_{1 \leqslant i_j \leqslant N_j}$, for all $j = 1, 2, \ldots, m$. $\mathcal{A}$ can be realized as the following set $\{(i_1, i_2, \ldots, i_m) : 1 \leqslant i_j \leqslant N_j, \forall j\}$, in which each point $(i_1, i_2, \ldots, i_m)$ is assigned the value $\alpha_{i_1 i_2 \ldots i_m}$. This realization of $\mathcal{A}$ is called an m-dimensional box-shape matrix of size $N_1 \times N_2 \times \cdots \times N_m$, where we associate with each such matrix a sub-ring $\mathrm{S}_{\mathcal{A}} = \mathbf{C}[\mathcal{A}] \subset \mathrm{S}$, where S is a commutative ring over the complex number field. For each $j = 1, 2, \ldots, m$, a two-by-two minor about the jth coordinate of $\mathcal{A}$ is given by

$$\mathcal{P}_{k_1 l_1; k_2 l_2; \ldots; k_m l_m}^{j} = \alpha_{k_1 k_2 \ldots k_m} \alpha_{l_1 l_2 \ldots l_m} - \alpha_{k_1 k_2 \ldots k_{j-1} l_j k_{j+1} \ldots k_m} \alpha_{l_1 l_2 \ldots l_{j-1} k_j l_{j+1} \ldots l_m} \in \mathrm{S}_{\mathcal{A}}. \tag{5}$$

Then the ideal $\mathcal{I}^m_{\mathcal{A}}$ of $S_{\mathcal{A}}$ is generated by $\mathcal{P}^j_{k_1l_1;k_2l_2;\ldots;k_ml_m}$ and describes the separable states in $\mathbf{CP}^{N_1N_2\cdots N_m-1}$ [8]. The image of the Segre embedding $\mathrm{Im}(\mathcal{S}_{N_1,N_2,\ldots,N_m})$, which again is an intersection of families of quadric hypersurfaces in $\mathbf{CP}^{N_1N_2\cdots N_m-1}$, is called the Segre variety and it is given by

$$\mathrm{Im}(\mathcal{S}_{N_1,N_2,\ldots,N_m}) = \bigcap_{\forall j} \mathcal{V}\big(\mathcal{P}^j_{k_1l_1;k_2l_2;\ldots;k_ml_m}\big). \tag{6}$$

Now, we define an entanglement measure for a pure multipartite state as

$$\begin{aligned}\mathcal{E}\big(\mathcal{Q}^p_m(N_1,\ldots,N_m)\big) &= \left(\mathcal{N}\sum_{\forall j}\big|\mathcal{P}^j_{k_1l_1;k_2l_2;\ldots;k_ml_m}\big|^2\right)^{\frac{1}{2}}\\ &= \left(\mathcal{N}\sum_{k_j,l_j,j=1,2,\ldots,m}\big|\alpha_{k_1k_2\ldots k_m}\alpha_{l_1l_2\ldots l_m}-\alpha_{k_1k_2\ldots k_{j-1}l_jk_{j+1}\ldots k_m}\alpha_{l_1l_2\ldots l_{j-1}k_jl_{j+1}\ldots l_m}\big|^2\right)^{\frac{1}{2}},\end{aligned} \tag{7}$$

where $\mathcal{N}$ is a normalization constant and $j = 1, 2, \ldots, m$. This measure coincides with the generalized concurrence for a general bipartite and three-partite state, but for reasons that we have explained in [4], it fails to quantify the entanglement for $m \geqslant 4$, whereas it still provides the condition of full separability. However, it is still possible to define an entanglement measure for general multipartite states if we modify equation (7) in such a way that it contains all possible permutations of indices. To do so, we propose a measure of entanglement for general pure multipartite states as

$$\begin{aligned}\mathcal{F}\big(\mathcal{Q}^p_m(N_1,\ldots,N_m)\big) = \Bigg(\mathcal{N}&\sum_{\forall\sigma\in\mathrm{Perm}(u)}\ \sum_{k_j,l_j,j=1,2,\ldots,m}\\ &\times\big|\alpha_{k_1k_2\ldots k_m}\alpha_{l_1l_2\ldots l_m}-\alpha_{\sigma(k_1)\sigma(k_2)\ldots\sigma(k_m)}\alpha_{\sigma(l_1)\sigma(l_2)\ldots\sigma(l_m)}\big|^2\Bigg)^{\frac{1}{2}},\end{aligned} \tag{8}$$

where $\sigma \in \mathrm{Perm}(u)$ denotes all possible sets of permutations of indices for which $k_1k_2\ldots k_m$ are replaced by $l_1l_2\ldots l_m$, and u is the number of indices to permute. By construction, this measure of entanglement vanishes on product states and it is also invariant under all possible permutations of indices. Note that the first set of permutations defines the Segre variety, but there are also additional complex projective varieties embedded in $\mathbf{CP}^{N_1N_2\cdots N_m-1}$ which are defined by other sets of permutations of indices in equation (8). These varieties are defined by similar quadratic polynomials as those used to define the Segre variety. As the Segre variety is defined by a completely decomposable tensor but our new varieties are defined by partially decomposable tensors, these varieties must be different. As an example we will discuss the four-partite system $\mathcal{Q}^p_m(N_1,\ldots,N_4)$, where we first encounter these new varieties. For this quantum system we can partition the Segre embedding as follows:

$$\begin{array}{ccc}\mathbf{CP}^{N_1-1}\times\mathbf{CP}^{N_2-1}\times\mathbf{CP}^{N_3-1}\times\mathbf{CP}^{N_4-1} & \xrightarrow{\mathcal{S}_{N_1,N_2}\otimes I\otimes I} & \mathbf{CP}^{M_1}\times\mathbf{CP}^{N_3-1}\times\mathbf{CP}^{N_4-1}\\ \Big\downarrow{\scriptstyle \mathcal{S}_{N_1,\ldots,N_4}} & & \Big\downarrow{\scriptstyle I\otimes\mathcal{S}_{N_3,N_4}}\\ \mathbf{CP}^{N_1N_2N_3N_4-1} & \xleftarrow{\mathcal{S}_{M_1,M_2}} & \mathbf{CP}^{M_1}\times\mathbf{CP}^{M_2}\end{array}$$

where $M_1 = N_1N_2 - 1$, $M_2 = N_3N_4 - 1$ and $(M_1 + 1)(M_2 + 1) = N_1N_2N_3N_4$. For the Segre variety, which is represented by a completely decomposable tensor, we have a commuting diagram and $\mathcal{S}_{N_1,\ldots,N_4} = \big(\mathcal{S}_{M_1,M_2}\big)\circ\big(I\otimes\mathcal{S}_{N_3,N_4}\big)\circ\big(\mathcal{S}_{N_1,N_2}\otimes I\otimes I\big)$. Now, if we assume that $\left(\mathcal{S}_{M_1,M_2}\right) \circ \left(I \otimes \mathcal{S}_{N_3,N_4}\right) \circ \left(\mathcal{S}_{N_1,N_2} \otimes I \otimes I\right)$ is not defined by a completely decomposable tensor, e.g., subsystems 1 and 2 are not decomposable, then this diagram does not commute. However, the image of the composite map $\left(\mathcal{S}_{M_1,M_2}\right) \circ \left(I \otimes \mathcal{S}_{N_3,N_4}\right) \circ \left(\mathcal{S}_{N_1,N_2} \otimes I \otimes I\right)$ gives a complex projective variety in $\mathbf{CP}^{N_1N_2N_3N_4-1}$, which is defined by this new variety and is not isomorphic to the Segre variety. This gives some insight about these new complex projective varieties. However, it could be interesting to further investigate the geometry of these complex projective varieties.

3. Some examples: three-partite and four-partite states

In this section, we will apply this measure of entanglement to three-partite and four-partite states and give explicit expressions for the measure of entanglement for these states. We start from the simplest multipartite states, namely three-partite states. Following equation (8), we can write the measure of entanglement for such states as

$$\begin{aligned}\mathcal{F}\big(\mathcal{Q}_3^p(N_1, N_2, N_3)\big) &= \Bigg(\mathcal{N} \sum_{\forall\sigma\in\mathrm{Perm}(u)} \sum_{k_j,l_j,j=1,2,3} \big|\alpha_{k_1k_2k_3}\alpha_{l_1l_2l_3} \\ &\quad - \alpha_{\sigma(k_1)\sigma(k_2)\sigma(k_3)}\alpha_{\sigma(l_1)\sigma(l_2)\sigma(l_3)}\big|^2\Bigg)^{\frac{1}{2}} \\ &= \Bigg(\mathcal{N}\sum_{p_1=1}^{3}\sum_{\forall k_j,l_j}\big|\alpha_{k_1k_2k_3}\alpha_{l_1l_2l_3} - \alpha_{k_1l_{p_1}k_3}\alpha_{l_1k_{p_1}l_3}\big|^2\Bigg)^{\frac{1}{2}} \\ &= \Bigg(\mathcal{N}\sum_{k_1l_1;k_2l_2;k_3l_3}\big(\big|\alpha_{k_1k_2k_3}\alpha_{l_1l_2l_3} - \alpha_{k_1k_2l_3}\alpha_{l_1l_2k_3}\big|^2 \\ &\quad + \big|\alpha_{k_1k_2k_3}\alpha_{l_1l_2l_3} - \alpha_{k_1l_2k_3}\alpha_{l_1k_2l_3}\big|^2\big) + \big|\alpha_{k_1k_2k_3}\alpha_{l_1l_2l_3} - \alpha_{l_1k_2k_3}\alpha_{k_1l_2l_3}\big|^2\Bigg)^{\frac{1}{2}}.\end{aligned} \tag{9}$$

This measure of entanglement for three-partite states (9) coincides with generalized concurrence [5] for quantum $\mathcal{Q}_3^p(N, N, N)$, that is whenever $N_1 = N_2 = N_3 = N$ and $\mathcal{N} = \frac{N}{6(N-1)}$. Moreover, this measure of entanglement is equivalent but not equal to our entanglement tensor based on joint positive operator valued measure on phase space [9]. For a three-qubit state $\mathcal{Q}_3^p(2, 2, 2)$, we have

$$\begin{aligned}\mathcal{F}\big(\mathcal{Q}_3^p(2,2,2)\big) &= \Bigg(\mathcal{N}\sum_{k_j,l_j=1,j=1,2,3}^{2}\big[\big|\alpha_{k_1k_2k_3}\alpha_{l_1l_2l_3} - \alpha_{l_1k_2k_3}\alpha_{k_1l_2l_3}\big|^2 \\ &\quad + \big|\alpha_{k_1k_2k_3}\alpha_{l_1l_2l_3} - \alpha_{k_1l_2k_3}\alpha_{l_1k_2l_3}\big|^2 + \big|\alpha_{k_1k_2k_3}\alpha_{l_1l_2l_3} - \alpha_{k_1k_2l_3}\alpha_{l_1l_2k_3}\big|^2\big]\Bigg)^{\frac{1}{2}} \\ &= (4\mathcal{N}\{2[|\alpha_{111}\alpha_{221} - \alpha_{121}\alpha_{211}|^2 + |\alpha_{112}\alpha_{222} - \alpha_{122}\alpha_{212}|^2 \\ &\quad + |\alpha_{111}\alpha_{212} - \alpha_{112}\alpha_{211}|^2 + |\alpha_{121}\alpha_{222} - \alpha_{122}\alpha_{221}|^2 + |\alpha_{111}\alpha_{122} - \alpha_{112}\alpha_{121}|^2 \\ &\quad + |\alpha_{211}\alpha_{222} - \alpha_{212}\alpha_{221}|^2] + |\alpha_{111}\alpha_{222} - \alpha_{112}\alpha_{221}|^2 + |\alpha_{111}\alpha_{222} - \alpha_{121}\alpha_{212}|^2 \\ &\quad + |\alpha_{111}\alpha_{222} - \alpha_{122}\alpha_{211}|^2 + |\alpha_{112}\alpha_{221} - \alpha_{121}\alpha_{212}|^2 + |\alpha_{112}\alpha_{221} - \alpha_{122}\alpha_{211}|^2 \\ &\quad + |\alpha_{121}\alpha_{212} - \alpha_{122}\alpha_{211}|^2\})^{\frac{1}{2}}.\end{aligned} \tag{10}$$

Next, we will discuss the measure of entanglement for four-partite states. In this case, we have more than one set of permutations, and as we have explained before this is the reason why the measure of entanglement that is directly based on the polynomial that defines the Segre variety fails to quantify the entanglement of four-partite states. Now, a measure of entanglement based on the modified Segre variety for four-partite states is given by

$$\mathcal{F}\big(\mathcal{Q}_4^p(N_1, N_2, N_3, N_4)\big) = \Bigg(\mathcal{N} \sum_{\forall\sigma\in\mathrm{Perm}(u)} \sum_{k_j,l_j,j=1,2,3,4} \big|\alpha_{k_1k_2k_3k_4}\alpha_{l_1l_2l_3l_4} - \alpha_{\sigma(k_1)\sigma(k_2)\sigma(k_3)\sigma(k_4)}\alpha_{\sigma(l_1)\sigma(l_2)\sigma(l_3)\sigma(l_4)}\big|^2\Bigg)^{\frac{1}{2}}$$
$$= \Bigg(\mathcal{N}\Bigg[\sum_{p_1=1}^{4}\sum_{\forall k_j,l_j} \big|\alpha_{k_1k_2k_3k_4}\alpha_{l_1l_2l_3l_4} - \alpha_{k_1k_2l_{p_1}k_4}\alpha_{l_1l_2k_{p_1}l_4}\big|^2 + \sum_{p_1<p_2}\sum_{\forall k_j,l_j} \big|\alpha_{k_1k_2k_3k_4}\alpha_{l_1l_2l_3l_4} - \alpha_{k_1l_{p_1}l_{p_2}k_4}\alpha_{l_1k_{p_1}k_{p_2}l_4}\big|^2\Bigg]\Bigg)^{\frac{1}{2}}. \tag{11}$$

An explicit expression for a four-qubit quantum system $\mathcal{Q}_4^p(2, 2, 2, 2)$ is given by

$$\mathcal{F}\big(\mathcal{Q}_4^p(2,2,2,2)\big) = \Bigg(\mathcal{N} \sum_{k_j,l_j=1,j=1,2,3,4}^{2} \Big[\big|\alpha_{k_1k_2k_3k_4}\alpha_{l_1l_2l_3l_4} - \alpha_{l_1k_2k_3k_4}\alpha_{k_1l_2l_3l_4}\big|^2$$
$$+ \big|\alpha_{k_1k_2k_3k_4}\alpha_{l_1l_2l_3l_4} - \alpha_{k_1l_2k_3k_4}\alpha_{l_1k_2l_3l_4}\big|^2 + \big|\alpha_{k_1k_2k_3k_4}\alpha_{l_1l_2l_3l_4} - \alpha_{k_1k_2l_3k_4}\alpha_{l_1l_2k_3l_4}\big|^2$$
$$+ \big|\alpha_{k_1k_2k_3k_4}\alpha_{l_1l_2l_3l_4} - \alpha_{k_1k_2k_3l_4}\alpha_{l_1l_2l_3k_4}\big|^2 + \big|\alpha_{k_1k_2k_3k_4}\alpha_{l_1l_2l_3l_4} - \alpha_{l_1l_2k_3k_4}\alpha_{k_1k_2l_3l_4}\big|^2$$
$$+ \big|\alpha_{k_1k_2k_3k_4}\alpha_{l_1l_2l_3l_4} - \alpha_{l_1k_2l_3k_4}\alpha_{k_1l_2k_3l_4}\big|^2 + \big|\alpha_{k_1k_2k_3k_4}\alpha_{l_1l_2l_3l_4} - \alpha_{l_1k_2k_3l_4}\alpha_{k_1l_2l_3k_4}\big|^2$$
$$+ \big|\alpha_{k_1k_2k_3k_4}\alpha_{l_1l_2l_3l_4} - \alpha_{k_1l_2l_3k_4}\alpha_{l_1k_2k_3l_4}\big|^2 + \big|\alpha_{k_1k_2k_3k_4}\alpha_{l_1l_2l_3l_4} - \alpha_{k_1l_2k_3l_4}\alpha_{l_1k_2l_3k_4}\big|^2$$
$$+ \big|\alpha_{k_1k_2k_3k_4}\alpha_{l_1l_2l_3l_4} - \alpha_{k_1k_2l_3l_4}\alpha_{l_1l_2k_3k_4}\big|^2\Big]\Bigg)^{\frac{1}{2}}. \tag{12}$$

The first set of permutations represented by the four first terms gives $4 \times 28 = 112$ quadratic terms and the second set of permutations, which is represented by the last six terms, gives $6 \times 6 = 36$ quadratic terms. Thus, the measure of entanglement for four-qubit states contains 148 terms. We suspect that the measure of entanglement for four-qubit states also coincides with the generalized concurrence [5] since both measures have the same number of terms and they are also constructed by quadratic polynomials. The generalized concurrence [5] is defined for composite quantum systems where all subsystems have the same dimension, namely $\mathcal{Q}_m^p(N, N, \ldots, N)$. However, our measure of entanglement works for any pure arbitrary dimensional quantum system $\mathcal{Q}_m^p(N_1, N_2, \ldots, N_m)$. Thus, it is possible that our measure of entanglement $\mathcal{F}\big(\mathcal{Q}_m^p(N_1, \ldots, N_m)\big)$ coincides with the generalized concurrence for the quantum systems $\mathcal{Q}_m^p(N, N, \ldots, N)$. In this case the normalization constant is given by $\mathcal{N} = \frac{N}{(2^m-2)(N-1)}$.

Acknowledgment

The author gratefully acknowledges the financial support of the Japan Society for the Promotion of Science (JSPS).

References

[1] Wootters W K 1998 *Phys. Rev. Lett.* **80** 2245
[2] Brody D C and Hughston L P 2001 *J. Geom. Phys.* **38** 19

[3] Miyake A 2003 *Phys. Rev.* A **67** 012108
[4] Heydari H and Björk G 2005 *J. Phys. A: Math. Gen.* **38** 3203–11
[5] Albeverio S and Fei S M 2001 *J. Opt. B: Quantum Semiclass. Opt.* **3** 223
[6] Griffiths P and Harris J 1978 *Principles of Algebraic Geometry* (New York: Wiley)
[7] Mumford D 1976 *Algebraic Geometry I, Complex Projective Varieties* (Berlin: Springer)
[8] Grone R 1977 *Proc. Am. Math. Soc.* **64** 227
[9] Heydari H and Björk G 2005 *Quantum Inform. Comput.* **5** 146–55